BASIC PROBLEMS OF A MICROSCOPIC THEORY OF A MANY BODY QUANTUM SYSTEM

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Basic problems of a microscopic theory of many body quantum systems and different aspects of a new approach which can help in solving them are discussed in detail. To this effect we make a critical study of the wave mechanics of two hard core quantum particles and discover its several untouched aspects, *viz.* : (i) the useful details of $\psi_k(r)$ (representing the relative motion of two particles), (ii) the expectation value of *hard core* (HC) repulsion ($< V_{HC}(r) >$), (iii) the inconsistency of the statements, $r \leq \sigma$ and $\psi_k(r \leq \sigma) = 0$ (σ = HC diameter of a particle), with uncertainty principle particularly for low k values, (iv) the lower bound of allowed values of $k = 2q$, (v) the dominance of interparticle phase correlation in low temperature phase. For the first time this study concludes that $< V_{HC}(r) >$ has *zero* value which does not agree with its *non-zero* value known for the last several decades. This also finds compelling reasons for a system of interacting bosons such as liquid ^4He to have (**q**, -**q**) pair condensation with allowed q, obviously controlled by $V_{HC}(r)$, to satisfy $q \geq \pi/d$. Several important aspects of N body quantum systems like liquids ^4He and ^3He are also concluded. Free from any error [see editor's note *J. Scientific Exploration* **16**(1), p.1 (2002)], our approach can help in developing nearly exact microscopic theories of widely different systems of interacting bosons and fermions, as demonstrated for liquids ^4He type systems [*J. Scientific Exploration*, **16**, 77-116 (2002)]. The paper also sums up the expert observations with our response to facilitate one to have a critical assessment and better understanding of the new approach.

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1.0 INTRODUCTION

Microscopic studies of a *system of interacting bosons* (SIB) such as liquid ^4He (LHe-4) and a *system of interacting fermions* (SIF) such as liquid ^3He (LHe-3) are of great fundamental importance [1, 2] because these systems provide an opportunity to investigate quantum effects at macroscopic level. The scope of such studies has been widened by the recent experimental discovery of Bose Einstein condensate (BEC) in trapped dilute gases because interparticle interactions in these gases too play as important role as in LHe-4 [3]. While the basic aspects of the subject are elegantly discussed in several texts [4, 5, 6, 7], the wealth of experimental and theoretical results generated from widely different studies of various SIB and SIF has been reviewed recently in several reports [8, 9], books [10, 11] and research articles [12, 13]. References to other important publications and review articles of older times can be traced from all these sources and our recent publications [14, 15, 16] presenting a new approach to the microscopic understanding of a SIB and its unification with that of a SIF.

It appears that theories of these systems, developed by using different mathematical tools and varying sets of assumptions and conjectures, can be placed into two groups, *viz.*, G1 group of the conventional theories which assume the existence of $p = 0$ condensate, $n_{p=0}(T)$ (a macroscopic fraction of particles condensed into a single particle state of $p = 0$), as the basic origin of superfluidity of a SIB [8, 10, 11] and the formation of a kind of Cooper pairs as the main factor for the superfluidity of a SIF [9, 10], and G2 group of a recently developed theoretical framework which uses a new approach to formulate the microscopic theories of a SIB [15] and SIF [17]. The approach makes no assumption like the existence of $p = 0$ condensate for a SIB or the formation of Cooper like pairs for a SIF. In stead it uses the results that we obtain by solving the N body Schrödinger equation. It concludes that (**q**, - **q**) bound pairs form an important basis for the unique properties of superfluid phase of a SIB/SIF [15, 16, 17].

In one of the two approaches of conventional theories, one starts with the Hamiltonian of the system written in terms of second quantized Schrödinger field and then proceeds to solve the problem by using several important inferences such as : (i) the HC potential can be used perturbatively by pseudo-potential method, (ii) coupling constant can be related directly to the two body scattering length (a), (iii) a dimensionless expansion parameter ($n.a^3$) can be used to perform perturbation calculation, *etc.* [1, 2, 3, 4, 5, 8]. However, in an alternative approach of variational calculations based on Jastrow correlation or Feenberg approximation one finds the radial distribution function, $g(r)$, and liquid structure factor, $S(Q)$, which can be used to calculate the *ground state* (G-state) properties, excitation spectrum, different thermodynamic properties and equation of state [12, 13]. Widely different methods of computer calculation of $g(r)$, $S(Q)$ and other properties are discussed and reviewed in [18]. In spite of highly complex mathematical formulations and calculations, these approaches are believed to have succeeded in obtaining certain results in support of

their presumptions of the existence of $p = 0$ condensate in superfluid ^4He and formation of (**q**, -**q**) bound pairs in case of ^3He as a basis of superfluidity of these systems. However, several experts in the field (*e.g.* Rickayzen [19], Woods and Cowley [20] and Sokol [21] in case of ^4He and Senatore and March [9] in case of ^3He) also underline the limitations and difficulties of these approaches in providing theories that explain the experimental properties of a SIB/SIF.

Our new approach, on the other hand, is a simple approach. It follows the standard method of solving the Schrödinger equation of the system and uses the wave mechanics of two HC particles as its basis. We make a critical study [14] of this mechanics to discover its several untouched aspects which serve as the basic foundations of our approach. We find [15, 16, 17] that : (i) each particle in a SIB/SIF represents (**q**, -**q**) pair moving with *center of mass* (CM) momentum **K**, (ii) with the onset of superfluid transition these particles form (**q**, -**q**) bound pairs with their phase positions locked at a relative separation of $\delta\phi = 2n\pi$; it happens with every particle representing a (**q**, -**q**) pair and leads the system to have a kind of collective binding and a macro-molecular behavior, (iii) while the λ-transition of a SIB represents the condensation of particles as (**q**, -**q**) pairs in the G-state of the system defined by $q = \pi/d$ and $K = 0$ [15], the superfluid transition of a SIF [16, 17] represents the fall of (**q**, -**q**) pairs in a state of $q = \pi/d$ and K ranging between $K = 0$ and $K = K_F$ (with K_F being the Fermi wave vector); the formation of (**q**, -**q**) bound pairs in both cases arises as a combined effect of interparticle phase correlations and interparticle attraction [15, 16, 17].

As demonstrated in [15], the new approach provides an almost exact theory of a SIB that explains the properties of liquid ^4He with unparalleled accuracy, simplicity and clarity. It has several advantages. It provides a framework that can unify the physics of widely different SIB (*e.g.* low dimensional systems, BEC state of dilute gases, *etc.* and SIF (*e.g.* superfluid ^3He, atomic nucleus, superconductors, *etc.*) [16, 17]. It does not require complex mathematical formulations and involved computer calculations. Evidently, these results provide enough reasons to search for the origin of the difficulties with our conventional approaches so that a comprehensive approach free from every problem can be identified for future studies. To facilitate the process we make a critical study of certain physical realities of LHe-4 anf LHe-3 systems and intricacies of wave particle duality and conclude that certain aspects of conventional approaches are in error and the new approach can help in developing an almost exact microscopic theory of these systems.

The paper has been arranged as follows. While Section 2.0 defines the basic nature of a SIB/SIF to which our approach can be applied, wave mechanics of two HC particles, which serves as an important basis of a theory of a SIB/SIF, is critically analyzed in Section 3.0. The way this analysis modifies our conventional understanding of a N body system is discussed in Section 4.0 and a summary of this study is presented in Section 5.0. While the paper is basically an expanded version of [14], it also analyzes some important aspects of N body systems for the first time. For example, by using a systematic mathematical derivation, this paper finds (*cf.* Section 4.5) that the expectation value of $V_{HC}(r_{ij})$ for every state of N body SIB/SIF has zero value. This has a direct conflict with non-zero value of $< V_{HC}(r_{ij}) >$ [4, 5], that we know for the last seven decades. Similarly the allowed q values satisfy $q \geq \pi/d$ indicating that no particle in a SIB/SIF has $q = 0$ as presumed in conventional formulations. Unfortunately, in spite of the fact that our approach provides a theory that explains experimentally observed properties of superfluid ^4He (even those which find no explanation in conventional theories), the accuracy and importance of our work has been ignored. Instead it faced unexpected difficulties with experts *not for any error* [22] but for the conflict of its basic ideas/results with those of their conventional framework. The history of the development of science witnesses many instances when non-conventional ideas/suggestions faced such difficulties before their acceptability. Consequently, we sum up all these difficulties and our response [23] so that the accuracy of different aspects of our work could be critically assessed and better understood. We hope that our approach should provide scientifically sound answers to different possible questions. It is satisfying that several competent people found merit in our work [22].

2.0 BASIC NATURE OF OUR SIB/SIF

To a good approximation, interparticle interaction in LHe-4/LHe-3 type system can be represented by two-body interaction $V(r_{ij})$ by presuming that the spin-spin type interactions (if exist) can be treated separately as a perturbation after solving the equation

$$\left[-\frac{\hbar^2}{2m} \sum_i^N \nabla_i^2 + \sum_{i<j} V(r_{ij}) \right] \Psi_n = E_n \Psi_n \qquad (1)$$

for its allowed states. $V(r_{ij})$ can be considered as the sum of : (i) the repulsive potential $V^{\rm R}(r_{ij})$ approximated to a HC interaction $V_{HC}(r_{ij})$ defined by $V_{HC}(r_{ij} \geq \sigma) = 0$ and $V_{HC}(r_{ij} < \sigma) = \infty$ with σ being the HC diameter of a particle and (ii) a relatively long range weak attraction $V^{\rm A}(r_{ij})$. Since $V^{\rm A}(r_{ij})$ can be replaced by constant negative potential (say, $-V_o$), atoms in these systems move freely like HC particles on a surface of potential $-V_o$. This not only agrees with the translation symmetry of a liquid [24], but also concludes that in formulating the theory of such a SIB/SIF we, at the first stage, have to deal with $V_{HC}(r_{ij})$ and $V^{\rm A}(r_{ij})$ can be used as a perturbation at the second stage. As such the effective N body

Hamiltonian, with which a first stage analysis should be started, can be expressed by

$$H^o(N) = -\frac{\hbar^2}{2m}\sum_i^N \bigtriangledown_i^2 + \sum_{i<j} V_{HC}(r_{ij}). \quad (2)$$

However, following a discussion in Section 4.8, it is evident that the effective interaction incorporated in our new approach hardly differs from the real interaction between the particles.

3.0 WAVE MECHANICS OF TWO HC PARTICLE

3.1. Schrödinger equation : In what follows from the above discussion, the wave mechanics of two particles (say P1 and P2) in a above defined SIB/SIF can be described by

$$\left[-\frac{\hbar^2}{2m}\left(\bigtriangledown_1^2 + \bigtriangledown_2^2\right) + V_{HC}(r)\right]\Psi(r_1, r_2) = E\Psi(r_1, r_2). \quad (3)$$

For a central force like $V_{HC}(r)$, Eqn.(3) can also be expressed [25] as

$$\left[-\frac{\hbar^2}{4m}\bigtriangledown_R^2 - \frac{\hbar^2}{m}\bigtriangledown_r^2 + V_{HC}(r)\right]\Psi(r, R) = E\Psi(r, R) \quad (4)$$

with

$$\Psi(r, R) = \psi_k(r)\exp\left[i(\mathbf{K}.\mathbf{R})\right] \quad (5)$$

describing a state of P1 and P2 having relative momentum $\mathbf{k}$ and CM momentum $\mathbf{K}$. Here $\psi_k(r)$ representing the relative motion of P1 and P2 satisfies

$$[-(\hbar^2/m)\bigtriangledown_r^2 + V_{HC}(r)]\psi_k(r) = E_k\psi_k(r) \quad (6)$$

with $E_k = E - \hbar^2K^2/4m$. Different notations in Eqns.(3-6) including

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \quad \text{and} \quad \mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1 \quad (7)$$

$$\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2 \quad \text{and} \quad \mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 \quad (8)$$

have their usual meaning. It is evident that while allowed $\mathbf{k}$ can be affected by $V_{HC}(r)$, similar values of $\mathbf{K}$ representing a kind of free particle motion ($\exp\left[i.(\mathbf{K}.\mathbf{R})\right]$) are controlled only by the dimensions of the container. This difference in the nature of $\mathbf{k}$ and $\mathbf{K}$ motions of a pair clearly indicates that these motions can get totally delinked under certain physical conditions of a SIB/SIF. In fact, as found in our recent study of a SIB [15], this really happens below λ-point and its superfluid phase is concluded to behave like a homogeneous mixture of two fluids as envisaged exactly by Landau [26].

3.2. Characteristic details of $\psi_k(r)$: The details of a $\psi_k(r)$, as defined by Eqns.(6 and 7), have so far been considered sufficient for most theoretical formulations of a SIB/SIF. However, we find no reason to ignore its additional details which not only simplify a theoretical formulation but also provide a better understanding of the system. To this effect we analyze $\psi_k(r)$ as seen from a frame attached to their CM defined by $\mathbf{R} = 0$ and $\mathbf{K} = 0$ and find that

$$\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{q} \quad (9)$$

and

$$\mathbf{r}_{CM}(1) = -\mathbf{r}_{CM}(2) \quad (10)$$

define the characteristic details of such a $\psi_k(r)$. Here $\mathbf{r}_{CM}$ and $\mathbf{q}$, respectively, represent the position and momentum of a particle with respect to the CM of the pair. It is evident that P1 and P2 in their relative dynamics have: (i) equal and opposite momenta ($\mathbf{q}$, -$\mathbf{q}$) (*cf.* Eqn.(9)) and (ii) they maintain a center of symmetry at their CM (*cf.* Eqn.(10)). In other words, two particles in a laboratory frame could be identified as a pair having ($\mathbf{q}$, -$\mathbf{q}$) momenta at their CM which by itself moves with momentum $\mathbf{K}$.

3.3. Equivalence of $V_{HC}(r)$ *and* $\delta(r)-$ *repulsion* : Note that two particles of momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ collide with each other at their CM defined by $r = 0$ and after this collision they either bounce back with an exchange of $\mathbf{k}_1$ and $\mathbf{k}_2$ or they appear to exchange their positions. This aspect of the dynamics of P1 and P2 does not change even if σ had been infinitely small (*i.e.* $\delta-$size). Only the probability of their collision should change with a change in σ and it would be higher for higher σ. Evidently, two colliding particles can be identified to behave effectively as particles of δ-size hard core or to interact through $\delta(r)$-repulsion. This implies $V_{HC}(r) \equiv A.\delta(r)$ where A representing the strength of $\delta(r)$-repulsion is given by [15] $A = h^2/8md^2$. Note that this equivalence does not differ significantly from $V_{HC}(r) \equiv A'.\delta(r)$ (with $A' = h^2.a/(\pi m.d^3) = (8a/\pi d).A$ and a = s-wave scattering length) obtained analytically [4, 5].

3.4. Correct form of $\Psi(r, R)$: In view of $V_{HC}(r) \equiv A.\delta(r)$, we find that two particles at $r \neq 0$ experience no interaction because $A.\delta(r)$ has zero value at all these points. Consequently, these particles can be represented by plane waves, $u_{k_i}(r_i) = (1/\sqrt{V}).\exp\left(i\mathbf{k}_i.\mathbf{r}_i\right)\exp\left[-iE_it/\hbar\right]$ and $\psi(r, R)$, at all $r \neq 0$, can be expressed either as

$$\Psi(r_1, r_2) = u_{k_1}(r_1)u_{k_2}(r_2) \quad (11)$$

when P1 and P2 having $r >> \lambda(\approx 2\pi/q)$ do not interfere with each other or as

$$\psi(r_1, r_2)^{\pm} = \frac{1}{\sqrt{2}}\left[u_{k_1}(r_1)u_{k_2}(r_2) \pm u_{k_2}(r_1)u_{k_1}(r_2)\right] \quad (12)$$

when they have $r \approx \lambda$. Eqn.(12) can be arranged to read as

$$\Psi(r,R)^{\pm} = \psi_k(r)^{\pm} \exp(i\mathbf{K}.\mathbf{R}) \exp[-i(E_k + E_K)t/\hbar] \tag{13}$$

with

$$\psi_k(r)^{+} = \sqrt{\frac{2}{V}} \cos(\mathbf{kr}/2). \exp[-iE_k t/\hbar] \tag{14}$$

and

$$\psi_k(r)^{-} = \sqrt{\frac{2}{V}} \sin(\mathbf{k}.\mathbf{r}/2) \exp[-iE_k t/\hbar]. \tag{15}$$

We find that $\psi_k(r)^+$ or $\psi_k(r)^-$ defines a kind of *stationary matter wave* (SMW) which modulates the relative phase position ($\phi = \mathbf{k}.\mathbf{r}$) of two particles in ϕ-space. As shown in [15], this modulation is the basic reason for the interparticle phase coherence experimentally observed in superfluid SIB and SIF; note that in optical lasers too we have cavity modes in the form of standing electromagnetic waves which modulate the phase positions of photons. Note that $\psi_k(r)^+$ and $\psi_k(r)^-$ differ only in terms of the origin of the $\phi = \mathbf{k}.\mathbf{r}$ scale by π. While $\phi = 0$ for $\psi_k(r)^+$ represents the center of an anti-nodal region of its SMW form, the same for $\psi_k(r)^-$ represents a nodal point. This concludes $\psi_k(r)^+ \equiv \psi_k(r)^-$. To understand this equivalence we note that in the wave mechanical superposition there is no way to find whether two particles after their collision have bounced back on their respective sides of their CM or they exchanged their positions across this point. Since the former case represents the self superposition of each particle, it can best be described by $\psi_k(r)^+$ because $V_{HC}(r)$ would not operate in such superposition. However, the latter case implying mutual superposition of two particles should be represented $\psi_k(r)^-$ because the corresponding wave function has to vanish at $r = 0$ due to $V_{HC}(r)$ operating between the two particles. One can also use $\psi_k(r)^-$ to represent the self superposition state of a particle with a understanding that the quantum spread of the particle starts from $r = 0$ (the nodal point of $\psi_k(r)^-$ identified with the CM of the pair) to $r = \lambda/2$ on the line joining the locations of P1 and P2. Evidently, $\Psi(r,R)^-$ is the best choice of the waveform that can correctly represent a pair of HC particles or *a particle as the part of this pair.*

3.5. Correct value of $< V_{HC}(r) >$: In view of what has been concluded in the above section, we have to use $\psi_k(r)^-$ (or $\Psi(r,R)^-$) to evaluate $< V_{HC}(r) >$ in a state of two HC particles (bosons and fermions alike). The anti-symmetry of $\psi(r,R)^-$ for the exchange of two particles, agrees with the observation of Woo [24] that HC bosons in configuration space behave the way fermions behave in momentum space. However, this does not mean that larger systems of HC bosons and HC fermions have no difference due to intrinsic spin values of bosons and fermions. They would differ in terms of the occupancy of the states of allowed K values. While any number of boson pairs can have same K, no two pairs of fermions have this liberty ; note that by using the relations, $\mathbf{k}_1 = \mathbf{K}/2 + \mathbf{q}$, $\mathbf{k}_2 = \mathbf{K}/2 - \mathbf{q}$, $\mathbf{k}'_1 = \mathbf{K}'/2 + \mathbf{q}'$ and $\mathbf{k}'_2 = \mathbf{K}'/2 - \mathbf{q}'$, one may find that $\mathbf{k}_1 \neq \mathbf{k}_2 \neq \mathbf{k}'_1 \neq \mathbf{k}'_2$ applicable to the set of four identical fermions is equivalent to $\mathbf{K} \neq \mathbf{K}'$ when $\mathbf{q} = \mathbf{q}'$ and $\mathbf{q} \neq \mathbf{q}'$ when $\mathbf{K} = \mathbf{K}'$. As such by using $\psi(r,R)^-$ (Eqn.13) as a state function of two HC particles we find that

$$< \psi(r,R)^- | A.\delta(r) | \psi(r,R)^- >= |\psi_k(r)^-|^2_{r=0} = 0 \tag{16}$$

which differs from its non-zero value reported in [4, 5]. It is evident that only one of the two values (zero or non-zero) of $< A.\delta(r) >$ can be true and one can easily find that Eqn.(16) is correct.

4.0. SOME BASIC RESULTS FOR N BODY SYSTEMS

4.1. Validity of Eqn.(3/4) for two HC *particles in a* SIB/SIF : Since Eqn.(3/4) basically describes two HC particles in free space, its validity for two such particles in a SIB/SIF must be established if we wish to use it as a basis to describe such a SIB/SIF. To this effect we note that two particles in a SIB/SIF encounter $V_{HC}(r)$ only when they collide with each other. While this elastic collision leads to an exchange of momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, the fact remains that P1 and P2 (before and after their collision) have free particle motion. Analyzing another possible situation in which mutually colliding P1 and P2 also collide with other particle(s), we find that $\mathbf{k}_1$ and $\mathbf{k}_2$ (or $\mathbf{k}$ and $\mathbf{K}$) after such collision may assume new values, $\mathbf{k}'_1$ and $\mathbf{k}'_2$ (or $\mathbf{k}'$ and $\mathbf{K}'$) but once again P1 and P2 retain their free particle motion. As such the interparticle interactions make a pair embedded in a SIB/SIF scatter/jump from its one state to another state of possible $\mathbf{k}$ and $\mathbf{K}$, while such a state of the pair in free space simply remains unchanged. Evidently, the basic nature of the dynamics of a pair in two situations does not differ which means that the states of P1 and P2 in such a system, in spite of their interaction with other particles, can be described by Eqn.(3/4).

4.2. Correct boundary condition : We find that the boundary condition, $\psi_k(r \leq \sigma) = 0$ or its equivalent, used to obtain a $\psi_k(r)$ either as a solution of Eqn.(6) or by way of a choice to construct a $\Psi(r,R)$, implicitly presumes that $r = \sigma$ can be determined precisely (*i.e.* with an uncertainty $\Delta r = 0$) which implies that momentum uncertainty Δk is infinitely large. However, since $\Delta k = \sqrt{(< k^2 > - < k >^2)}$ for the pair in $\psi_k(r)^-$ (Eqn.15) state can be at the most equal to k [27], $\Delta r = 0$ would be inconsistent with uncertainty principle for finite k and we have no pair of particles in a SIB/SIF having $k = \infty$. The degree of this inconsistency assumes

prominence, particularly, for particles of low momentum, *viz.*, $q < \pi/\sigma$ (*i.e.* $\lambda/2 > \sigma$) for which uncertainty in the positions of each particle, $\Delta r = \lambda/2$, becomes much larger than r. Evidently, one needs to find a right alternative of $\psi_k(r \leq \sigma) = 0$ condition. In this context we take cognizance of the fact that a particle in wave mechanics manifests itself as a *wave packet*(WP) of size $\lambda/2$ (*i.e.* a *sphere* of diameter $\lambda/2$) and because two HC particles do not overlap, their representative WPs should also have no overlap. Thus, the separation ($< r >$) between two particles should satisfy $< r > \geq \lambda/2$ condition or $k < r > \geq 2\pi$) [27] which also follows from the uncertainty relation, $\Delta k \Delta r \geq 2\pi$, because we expect $k \geq \Delta k$, and $< r > \geq \Delta r$.

4.3. Allowed values of q : Since two particles in a LHe-4/LHe-3 have shortest $< r >= d$ (the average nearest neighbor separation), $< r > \geq \lambda/2$ could be read as $d \geq \lambda/2$ which implies that lower bound of q should be $q = q_o = \pi/d$. This does not support the assumption that particles in a LHe-4/LHe-3 can have $q = 0$ and there can exist a $q = 0$ condensate $n_{p=0}(T)$ in LHe-4 below T_λ. This inference agrees with the fact that the existence of $n_{p=0}(T)$ in superfluid ^4He has not been proved experimentally beyond doubt [21]. In fact, as discussed above, this assumption contradicts the uncertainty relation. The lowest possible $q = q_o = \pi/d$ and corresponding energy $\varepsilon_o = h^2/8md^2$, so inferred, imply that each particle in the system exclusively occupies a cavity of volume d^3 and agrees with excluded volume condition [28] supposed to be satisfied by HC particles.

4.4. Interparticle phase correlations : When a particle manifests itself as a wave, its phase position ϕ becomes more relevant than its r. Naturally, in the low temperature phase of a SIB/SIF where wave nature dominates the particle nature, the interparticle phase correlation defined by $g(\phi) = |\Psi(r,R)^-|^2 = |\psi_k(r)^-|^2$ (an explicit consequence of wave nature) should be more relevant than interparticle position correlation defined by $g(r) = |\phi_k(r)|^2$ (a consequence of interparticle potential) where $\phi_k(r)$ is the solution of hyper-netted chain Schrödinger equation [12, 13]. The fact that this inference also applies to a LHe-3 is well evident since the details of $\psi_k(r)^-$ describing relative dynamics of two particles does not differ for a pair of HC bosons and that of fermions. $g(\phi)$ can also be expressed by an equivalent quantum correlation potential [15] $U = -k_B T_o . \ln|\psi_k(r)^-|^2 = -k_B T_o . \ln 2\sin^2(\mathbf{k}.\mathbf{r}/2)$ (with T_o being the temperature equivalent of ε_o) having minimum ($= -k_B T_o . \ln 2$) and maximum ($=\infty$) periodically located at $\phi = \mathbf{k}.\mathbf{r} = (2n+1)\pi$ (midpoint of an anti-nodal region a SMW) and $2n\pi$ (location of a node of SMW), respectively. As such this potential locks the particles at phase position separation $\Delta\phi = 2n\pi$ (with $n = 1, 2, 3, ...$) and explains the origin of the experimentally observed phase coherence in the motion of particles in a superfluid phase of a SIB/SIF. It is important to note that our emphasis on $g(\phi)$ does not ignore $g(r)$. Rather it gives equal importance of $g(r)$ and $g(k)$ (the interparticle momentum correlation) because of $\phi = \mathbf{k}.\mathbf{r} = 2n\pi$, while the emphasis on only $g(r)$ ignores $g(k)$.

4.5. $< \Phi_n|V_{HC}(r)|\Phi_n >$: As reported in [4, 5], one obtains a non-zero $< \Phi_n|V_{HC}(r) \equiv A.\delta(r)|\Phi_n >$ when he uses

$$\Phi_n = \frac{1}{\sqrt{N!}} \sum_P (\pm 1)^{[P]} [u_{P\alpha_i}(r_1) u_{P\alpha_j}(r_2) ... u_{P\alpha_N}(r_N)] \tag{17}$$

to represent a state of N particles and evaluates the integral over r_1 and r_2. However, in this process he does not encounter $\psi_k(r)^-$ to implement $\psi_k(r)^-|_{r=0} = 0$, -the basic character of a pair waveform. Naturally, this process does not ensure the accuracy of non-zero $< \Phi_n|V_{HC}(r)|\Phi_n >$ so obtained [4, 5]. Guided by this observation, we rearrange Eqn.(17) as

$$\begin{aligned} \Phi_n &= \frac{1}{\sqrt{N!}} \sum_{i<j} \left\{ u_{\alpha_i}(r_1) u_{\alpha_j}(r_2) \pm u_{\alpha_j}(r_1) u_{\alpha_i}(r_2) \right\} . \\ &\quad \sum_{P'} (\pm 1)^{[P']} [u_{P'\alpha_l}(r_3) ... u_{P'\alpha_N}(r_N)] \end{aligned} \tag{18}$$

which resolves $\psi_k(r)^-$ and $\exp(i\ \mathbf{K}.\ \mathbf{R})$ waveforms of two particles at r_1 and r_2 (say) and evaluate the $< \Phi_n|V_{HC}(r)|\Phi_n >$ related integral over r and R. Note that P' in Eqn.(18) represents the permutation over different α includes all N α (*i.e.* α_1, α_2, $...\alpha_N$) except those identified as α_i and α_j (with integers i and j running from 1 to N). Defining

$$\begin{aligned} \psi_{\alpha_i,\alpha_j}(r,R)^\pm &= [u_{\alpha_i}(r_1) u_{\alpha_j}(r_2) \pm u_{\alpha_j}(r_1) u_{\alpha_i}(r_2)] \\ &= \psi_{\alpha_i,\alpha_j}(r)^\pm \exp[i(\mathbf{k}_i + \mathbf{k}_j).\mathbf{R}] \end{aligned} \tag{19}$$

we find that

$$< \Phi_n|\delta(r)|\Phi_n >= \frac{N(N-1)}{2} I^{(1)}_{ij,i'j'} . I^{(2)}_{ij,i'j'} . I^{(3)} \tag{20}$$

with

$$\begin{aligned} I^{(1)}_{ij,i'j'} &= \int \psi_{\alpha_i,\alpha_j}(r)^c \delta(r) \psi_{\alpha'_i,\alpha'_j}(r)^c d^3 r \\ &= \psi_{\alpha_i,\alpha_j}(r)^c|_{r=0} . \psi_{\alpha'_i,\alpha'_j}(r)^c|_{r=0} = 0. \end{aligned} \tag{21}$$

Here $\psi_{\alpha_i,\alpha_j}(r)^c$ is that $\psi_{\alpha_i,\alpha_j}(r)^\pm$ which vanishes at $r = 0$ since only such a waveform can correctly represent two HC particles (bosons/fermions). We also have

$$\begin{aligned} I^{(2)}_{ij,i'j'} &= \int \exp[-i(\mathbf{k}_i + \mathbf{k}_j).\mathbf{R}] \\ &\quad . \exp[i(\mathbf{k}_{i'} + \mathbf{k}_{j'}).\mathbf{R}] d^3 R \\ &= \delta_{(\mathbf{k}_i + \mathbf{k}_j),(\mathbf{k}_{i'} + \mathbf{k}_{j'})} \end{aligned} \tag{22}$$

representing the conservation of momentum of two particles in their collision, and

$$\begin{aligned} I^{(3)} =< \Phi'_n|\Phi'_n > &= \frac{1}{(N-2)!} \sum_{P'} \sum_{Q'} (\pm 1)^{[P']+[Q']} \\ &\quad [\delta_{P'\alpha_3,Q'\alpha_3} ... \delta_{P'\alpha_N,Q'\alpha_N}] \end{aligned} \tag{23}$$

where Φ'_n, -a part of Eqn.(18), can be expressed as

$$\Phi'_n(r_3,...,r_N) = \frac{1}{\sqrt{(N-2)!}}\sum_{P'}(\pm 1)^{[P']}[u_{P'\alpha_3}(r_3) u_{P'\alpha_4}(r_4)...u_{P'\alpha_N}(r_N)]. \quad (24)$$

Evidently, $< \Phi_n|A.\delta(r)|\Phi_n >$ vanishing due to zero value of $I^{(1)}_{ij,i'j'}$ (Eqn.21) renders

$$< \Phi_n|A.\sum_{a<b}\delta(r_b - r_a)|\Phi_n >= 0 \quad (25)$$

which differs from its non-zero value [4, 5]. It is important to note that the zero value of $< \Phi_n|A.\delta(r)|\Phi_n >$ should not be confused to imply that energy eigenvalues of a system of non-interacting particles are identical to those of a system of particles interacting through $V_{HC}(r)$. While particles in the former case have no way to identify the presence of each other and the lowest possible $q = q_o = \pi/L$ and corresponding energy $\varepsilon_o = h^2/8mL^2$ are decided by the size L of the container, q_o and ε_o, in the latter case (*cf.* Section 4.2), are decided by d which is not only much shorter than L but is also decided by net sum of $V(r_{ij})$.

4.6. Energy eigenvalue : It is evident that a particle in a SIB/SIF can be represented more accurately by a $\Psi(r,R)^-$ pair waveform (Eqn.13) than a plane wave due to its wave mechanical superposition with itself (self superposition) or with another particle (mutual superposition) in a process of its collision; self superposition of a particle can also be visualized when it returns after its collision with the walls of the container. However, the consistency of this representation demands that the kinetic energy terms of $H^o(N)$ (Eqn.2) should also be paired as

$$h(i) = \frac{1}{2}[h_i + h_{i+1}] \quad (26)$$

with $h_{N+1} = h_1$ and

$$h_i = -\frac{\hbar^2}{2m}\nabla_i^2 \quad \text{and} \quad h(i) = -\frac{\hbar^2}{8m}\nabla^2_{R_i} - \frac{\hbar^2}{2m}\nabla^2_{r_i} \quad (27)$$

Naturally, this renders

$$H^o(N) = \sum_i^N h(i) + \sum_{i<j}^N A.\delta(r_{ij}). \quad (28)$$

One can also use other schemes of pairing h_i terms of $H^o(N)$ as discussed in [15]. The rearrangement of $H^o(N)$ as expressed by Eqn.(28) and representation of each particle by a $\Psi(r,R)^-$ type pair waveform facilitate in constructing a wave function for a state of N particle system by following the standard procedure. This is particularly so because expectation value of $V_{HC}(r_{ij}) \equiv A.\delta(r_{ij})$ vanishes (*cf.* Eqns. 16 and 25). For N particles we have N different $\Psi(r,R)^-$ rendering $\Sigma = N!$ different Ψ_n (through permutations of q_i or r_i of N particles) for $n-$th state of equal energy $E_n = E_n(K) + E_n(k)$. Defining

$$E_n(K) = \sum_i^N \varepsilon(K)_i \quad \text{and} \quad E_n(k) = \sum_i^N \varepsilon(k)_i \quad (29)$$

we have

$$\Psi_n = \phi_n(q).\phi_n(K) \quad (30)$$

with

$$\phi_n(q) = \left[\left(\frac{2}{\mathrm{V}}\right)^{\frac{N}{2}}\prod_{i=1}^N \sin(\mathbf{q}_i.\mathbf{r}_i)\right]\exp[-iE_n(k)t/\hbar] \quad (31)$$

and

$$\phi_n(K) = A.\left(\frac{1}{\mathrm{V}}\right)^{\frac{N}{2}}\sum_{pK}(\pm 1)^p\prod_{i=1}^N \exp[i(\mathbf{K}_i.\mathbf{R}_i]. \exp[-iE_n(K)t/\hbar] \quad (32)$$

with $A = \sqrt{1/N!}$. Here $\sum_{pK}(\pm 1)^p$ refers to the sum of different permutations of K over all particles. While the use of $(+1)^p$ or $(-1)^p$ in Eqn.(32) depends on the bosonic or fermionic nature of the system for their spin character of the particles, the use of the restriction $q_i \geq \pi/d$ in Eqn.(31) treats the so called fermionic behavior (in the $r-$space) of HC particles (bosons and fermions alike). Evidently, a state function of N HC bosons should differ from that of N HC fermions in the choice of $(+1)^p$ or $(-1)^p$. Note that Σ different Ψ_n counted above take care of the permutation of $k = 2q$. We have

$$\Phi_n = \frac{1}{\sqrt{\Sigma}}.\sum_i^\Sigma \Psi_n^{(i)} \quad (33)$$

which represents the general form of a state function that should reveal the physics of a SIB/SIF. Note that Φ_n represents a state where each particle, as a WP of size $= \pi/q$, has a plane wave motion of momentum $\mathbf{K}$. Since $\phi_n(q)$ (Eqn.31) appearing in each $\Psi_n^{(i)}$ of Eqn.(33) through Eqn.(30) vanishes at every point $r_i = r_j$, that defines the CM of $i-$th and $j-$th particles, one may easily find

$$< \Phi_n|V_{HC}(r_{ij}) \equiv A.\delta(r_{ij})|\Phi_n >= 0 \quad (34)$$

and

$$< \Phi_n|H^o(N)|\Phi_n > = \sum_i^N \frac{\hbar^2}{8m}\left(K_i^2 + k_i^2\right) = \sum_i^N \frac{\hbar^2}{8m}\left(K_i^2 + 4q_i^2\right) \quad (35)$$

which can be shown [15] to render

$$E_o = Nh^2/8md^2 = N\varepsilon_o \tag{36}$$

as the G-state energy for a SIB for which all K values can be zero. However the G-state energy for a SIF, where only two particles (with spin up and spin down) can occupy a state of an allowed K, is found to be [16, 17]

$$E_o = N\varepsilon_o + E_F \tag{37}$$

with $E_F = h^2/8\pi m(N/1.5045\mathrm{V})^{2/3}$ representing the Fermi energy of N particles derived for their K motions. Note that K motion of a particle in SMW configuration is a kind of free particle motion with effective mass being $4m$.

4.7. $V^{\mathrm{A}}(r_{ij})$ *as perturbation* : It is obvious that $V^{\mathrm{A}}(r_{ij})$ can affect the relative configuration of particles in a SIB/SIF which, in the G-state of the system, is defined by the nearest neighbor separation d and the least possible q of a particle $q_o = \pi/d$. Evidently, when this state of the system is subjected to a perturbation by attractive potential, $V^{\mathrm{A}}(r_{ij})$, its G-state energy $E_o = N\varepsilon_o = Nh^2/8md^2$ changes to $E'_o = N\varepsilon'_o = Nh^2/8md'^2$ rendering a net fall in G-state energy by

$$E_g = \frac{Nh^2}{8m}\left[\frac{1}{d^2} - \frac{1}{d'^2}\right] \approx \frac{Nh^2}{4m}\frac{d'-d}{d^3} \tag{38}$$

where d' represents the increased d value which arises due to zero point repulsion coming into effect at a temperature at which particles in the system satisfy $\lambda_T = 2d$ (with thermal de Broglie wave length, $\lambda_T = h/\sqrt{2\pi m k_B T}$). In fact this force leads the system to have volume expansion on its cooling below certain temperature as observed experimentally for LHe-4 as well as LHe-3. We define E_g as the energy gap between the normal and superfluid states of the system. It also represents a kind of collective binding of all atoms for which the entire system can be identified as a single macro-molecule.

4.8. Effective interparticle interaction : It may be noted that our approach of developing the microscopic theory not only replaces $V^{\mathrm{R}}(r_{ij})$, as an approximation, by $V_{HC}(r_{ij}) \equiv A.\delta(r_{ij})$ but also imposes a condition, "that two WPs of HC particles should not share any point $\mathbf{r}$ in configuration space", -equivalent to assuming the presence of a repulsion of *finite range*, $r_a = \lambda/2$; it appears that the WP manifestation of particle extends the range of the influence of $V_{HC}(r)$ from $r_a = \sigma$ to $r_a = \lambda/2$ when $\lambda/2 > \sigma$. This repulsion is nothing but the zero point repulsion [29] which can be derived as the first d derivative of $\varepsilon_o = h^2/8md^2$ representing the G-state energy of a particle. This again shows that A in $V_{HC}(r) \equiv A.\delta(r)$ should be $h^2/8md^2$.

Further, since $V^{\mathrm{R}}(r_{ij})$ in most SIB falls faster (in LHe-4 it varies as r^{-12}) than the zero-point repulsion, varying as r^{-2}, the latter would dominate $V^{\mathrm{R}}(r_{ij})$ particularly for all $r > \sigma$ and $\leq \lambda/2$ and this observation agrees with the experimental facts that : (i) LHe-4 and LHe-3 do not solidify due to zero-point repulsion even at $T = 0$ unless they are subjected to an external pressure of ≈ 25 and ≈ 30 atms, respectively, and (ii) they exhibit volume expansion with falling T around 2.2 and 0.5 K, respectively[30]. It may also be noted that our condition, $\lambda/2 \leq d$, identifies d as the upper limit of the WP size $\lambda/2$ (the key aspect of our theory) of a particle and d is decided by the net sum of $V(r_{ij})$ without any approximation. Evidently, these observations prove that *our theory accounts for the* $V^{\mathrm{R}}(r_{ij})$ *and* $V^{\mathrm{A}}(r_{ij})$ *components of* $V(r_{ij})$ *close to their real effect.*

*4.9. Why (**q**, -**q**) pair condensation* : The phenomenon of superfluidity/superconductivity of a fermionic system is attributed to the condensation of Cooper pairs of fermions for a reason that the Pauli exclusion principle forbids two identical fermions from occupying single energy state, while any number of these pairs presumed to behave like bosons can do so. Because Pauli exclusion does not apply to bosons, conventional theorists find no difficulty in assuming the condensation of macroscopically large number of bosons into a single particle state of $p = 0$ as their main theme. However, this assumption ignores the fact that the way two fermions do not occupy same point in $k-$space, two HC particles do not occupy same point in $r-$space. This is particularly important because the requirement of antisymmetry of two fermion wave function, $\Phi_a(1,2) = [v_{\mathbf{k}'}(\mathbf{r}_1).v_{\mathbf{k}''}(\mathbf{r}_2) - v_{\mathbf{k}'}(\mathbf{r}_2).v_{\mathbf{k}''}(\mathbf{r}_1)]$, for their exchange, makes $\Phi_a(1,2)$ vanish not only for $\mathbf{k}' = \mathbf{k}''$ but also for $\mathbf{r}_1 = \mathbf{r}_2$. Evidently, if $\Phi(1,2)$ of two HC particles is subjected to a condition that it should vanish for $\mathbf{r}_1 = \mathbf{r}_2$, $\Phi(1,2)$ has to be identically antisymmetric and would, obviously, vanish also for $\mathbf{k}' = \mathbf{k}''$. This implies that two HC quantum particles in $r-$space behave like two fermions behave in $k-$space and concludes that two HC particles (*excluded to have* $\mathbf{r}_1 = \mathbf{r}_2$) can not have $\mathbf{k}' = \mathbf{k}''$, particularly, in a state of their *wave mechanical superposition* (*i.e.* a quantum state of $\lambda > d$). Note that the inference would be valid not only for particles of $\sigma \approx 0$ (*i.e.* particles interacting through δ-function repulsion) but also for ^4He type atoms because finite size HC repulsion becomes equivalent to δ-function repulsion for particles of $\lambda/2 > \sigma$ [14]. However, we also note that there is a difference in Fermi behavior due to HC nature and that due to half integer spin; while the former excludes every particle from having $q < \pi/d$ (applies identically to HC bosons and HC fermions), the latter excludes two particles (applies to fermions only) from having equal $\mathbf{K}$. Evidently, this excludes the possibility of nonzero $n_{p=0}(T)$ which has also been shown to be *inconsistent with excluded volume condition of* HC *particles* [28].

As such like Pauli exclusion provides effective repulsion to keep two fermions apart [20], the *volume exclusion condition applicable to* HC *quantum particles* and

WP *manifestation of quantum particles render such repulsion to keep their* WPs *at* $r \geq \lambda/2$; experimentally observed volume expansion of LHe-4 with decreasing T near T_λ [30] corroborates this fact. Evidently, there is no doubt that superfluidity of LHe-4 type SIB originates from the condensation of (**q**, -**q**) pairs. The *binding between two particles originates from their inherent interatomic attraction* and this has been discussed in detail in Section (5.4) of [15].

5.0. CONCLUSION

As such, this study concludes that : (i) Under the approximation $V_{HC}(r) \equiv A.\delta(r)$, $\Psi(r_1, r_2)$ (Eqn. 11) (for the situation $r > \lambda/2$) and $\psi_k(r, R)^-$ (Eqn. 13) (for $r \approx \lambda/2$) represent exact solutions of Eqns.(3) and (4), respectively, (ii) allowed q values satisfy $q \geq \pi/d$ (*cf.* Section 4.3), (iii) ϕ-correlations dominates the behavior of low temperature phase (*cf.* Section 4.4), and (iv) $< \Phi_n|A.\delta(r)|\Phi_n >$ has zero value (*cf.* Section 4.5) which differs significantly from its non-zero value concluded through conventional formulations[4, 5]. The errors of non-zero value of $< \Phi_n|A.\delta(r)|\Phi_n >$ are also evident from its dependence on momentum and spin distribution of particles[4, 5] which does not agree with the fact that $A.\delta(r = 0) = \infty$ and $A.\delta(r > 0) = 0$ is independent of momentum and/or spin states of two HC particles. As such the main sources of our basic problems in developing a correct theory of a LHe-4/LHe-3 have been the finer intricacies of wave mechanics (*cf.* Sections. 4.2-4.6 and Eqns.(16 AND 25) that we, somehow, missed for so long. This author too never imagined of this fact. Only recently, we identifies these intricacies and developed a new approach that resolves these problems and helps in finding an almost exact theory of a LHe-4(LHe-3) type SIB(SIF)[15, 16, 17]. As such the present analysis provides strong foundations to our approach. We also find that : (i) our approach not only uses $V^{\rm R}(r_{ij})$ and $V^{\rm A}(r_{ij})$ components of $V(r_{ij})$ effectively but also analyzes the way zero-point repulsion dominates the behavior of a superfluid SIB (*cf.* Section (4.8)) and (ii) there are compelling reasons (*cf.* Section (4.9)) for a SIB too to have (**q**, -**q**) pair condensation.

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[22] The editorial notes (J. Scientific exploration **16**(1), 1 (2002)) based on the opinion of several competent people and a thorough review of our work by a highly respectable theoretical physicist underlines the fact that our work [14, 15, 16] has no error.

[23] Two of their difficulties that (i) our approach considers only HC interactions and (ii) there are no compelling reasons for a SIB to have (**q**, -**q**) pair condensation, which do not need reference to the details of our theory [15], can not be accepted in view our response in Sections 4.8 and 4.9 of this paper and Sections 2.7 and 2.8 of [15]. Two other difficulties that (iii) the origin of bound pair formation needs to be explained in detail and (iv) the theory should also explain $\lambda-$transition in terms of a free energy expressed as a function of some suitable order parameter are more like suggestions. These were accepted and necessary additions, as given in Section 5.4 and Appendix -A of [15], respectively, were made. Unfortunately, their fifth difficulty against our emphasis on interparticle phase correlation $g(\phi)$ (in place of their emphasis of on interparticle position correlation $g(r)$) as an important basis for the low temperature properties remained a serious difference between us. However, it is evident that the accuracy of our emphasis on $g(\phi)$ (*cf.* Sections 4.4) should not be questioned because it is an obvious property of $\psi_k(r)^-$ and it is supported not only by scientifically sound logic and arguments but also by experimentally observed coherence in the relative motion of particles.

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